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Generalization of extensive structures and its representation

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HIGHLIGHTS

- A generalization of extensive structures and its representation are considered.
- A left nonnegative concatenation structure with left identity is defined.
- This structure satisfies solvability and Archimedeaness with left-concatenation.
- Two conditions make the structure into an extensive structure with identity.
- We get the weighted additive model as a representation on the extensive structure.

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ABSTRACT

This paper generalizes extensive structures so that a weighted additive model can be obtained. A left nonnegative concatenation structure with left identity is defined as a nonnegative concatenation structure (Luce et al., 1990) with left identity for which the solvability and Archimedean properties are satisfied only related to left-concatenation. This structure has two partial binary operations – multiplication and right division – and a new partial binary operation is defined on it. Two conditions of equivalence form are then provided to make the left nonnegative concatenation structure with left identity into an extensive structure with identity with respect to the newly defined operation. Finally, the weighted additive model is derived from an additive representation on the extensive structure, so that distinct m-period and n-period ($m \neq n$) temporal sequences can be compared.

1. Introduction

Matsushita (2011) recently generalized the classical result of Hölder (1901) in the context of groupoids (a "groupoid" is a nonempty set with a binary operation), and developed an axiom system to construct a weighted additive model. From groupoid multiplication, let ab denote the concatenation of commodities a, b. Then his model is of the following form:

$$u(ab) = \alpha u(a) + u(b), \quad \alpha \geqslant 1.$$

The first aim of this paper is to convert his algebraic axioms into a decision-making version so that they can be empirically tested. Meanwhile, all axioms, including the remaining ones, are to be rewritten under the requirement that the multiplication be generalized to a partial binary operation, that is, a generalization of extensive structures. Although the framework for constructing the weighted additive model is almost identical to the proof of Theorem 4.2 (Matsushita, 2011), the addition of some mathematical work is needed to achieve this aim. First, two axioms A8 and A9 (Lemma 1), written in a simple form, are proposed from which one can deduce the algebraic axioms. Second, the concepts of extensive "substructure" and "order-isomorphism" (Lemma 3) are introduced to yield the multiplicative form $\alpha u(a)$ in the weighted additive model.

We shall now consider preferences over temporal sequences of amounts of money. Many people will probably prefer receiving \$10,000 this year and \$5000 next year to receiving \$5000 this year and \$10,000 next year. A major reason for this preference is that the value of commodities decreases with the passage of time. Utility models has been already proposed to explain this kind of preference. The simplest one is of the following form: letting $(a_1, \ldots, a_n)$ denote an n-period temporal sequence,

$$\phi(a_1, \ldots, a_n) = \sum_{i=1}^{n} \lambda^{i-1} v(a_i),$$

where ϕ and v are real-valued functions on the set of temporal sequences consisting of n commodities and on the set of single commodities, respectively, and $\lambda \leqslant 1$ is a discount factor at a constant rate.

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The axiomatization to construct this utility model was initiated by Krantz, Luce, Suppes, and Tversky (1971) and Fishburn (1970). They developed a utility model with various discount factors so as to reflect the concept for a preference for advancing the timing of future satisfaction (i.e., impatience; Koopmans, 1960; Koopmans, Diamond, & Williamson, 1964). Then incorporating "stationarity[1]" by Koopmans (1960), they reduced the utility model to the above special model with a discount factor at a constant rate. For this construction, Krantz et al. assumed an "additive conjoint structure" and Fishburn considered a finite product of topological spaces. As such, the following problem arose: comparisons could be made only between temporal sequences with the same number of periods. Further, some of their axioms are difficult to empirically test. Indeed, the n-factor independence condition requires us to consider the ordering of the joint effect of multiple factors in verifying its validity; the validity of the topological conditions (connectedness, separability) is, in itself, nearly impossible to directly test, because it is difficult to have subjects recognize the concept of open or closed sets in the frame of a preference structure.

Our weighted additive model (displayed in the first paragraph) too can deal with multi-period temporal sequences. Identifying $(a_1, \ldots, a_n)$ with $(\cdots((a_1a_2)a_3)\cdots a_{n-1})a_n$, from the inductive use of the equation of the weighted additive model, we have $u[(\cdots((a_1a_2)a_3)\cdots a_{n-1})a_n] = \sum_{i=1}^{n} \alpha^{n-i}u(a_i)$. It should be noted that this is a representation for multiplication. Since every temporal sequence (consisting of any number of commodities) is expressed as a product, this model can numerically evaluate preferences between distinct m-period and n-period ($m \neq n$) temporal sequences. This is a great advantage of our model over the above utility model with a stationary discount factor. Furthermore, in connection with the first aim of the paper, the axioms are to be written as equivalences between commodities or concatenations so that their validity can be empirically tested. Thus, the axiomatization of our weighted additive model offers a solution to the problems raised above. Another marked difference between these two models is that the weight of our model is $\alpha \geqslant 1$, which may be referred to as a markup factor at a constant rate. However, the concepts of a discount factor and a markup factor could be deemed relative, because for one temporal sequence, the receipt of each component is considered postponed or advanced depending on whether one is regarding the oldest or the latest period as a standard; and whether a utility model has a discount factor or a markup factor is determined on the basis of whether one counts each period number in the temporal sequence toward the future direction or toward the past direction. As such, our utility model can explain a preference property, such as impatience. From the above, the second aim of the paper is to put an interpretation on several axioms in the context of the decision-making problems of temporal sequences.

The rest of this paper is organized as follows. Section 2 provides the axioms to define a basic structure, called left nonnegative concatenation structure with left identity, the positive part of which is a generalized concept of a PCS (Luce, Krantz, Suppes, & Tversky, 1990) in the sense that the solvability and Archimedean properties are satisfied only related to left-concatenation. Moreover, some properties are shown to be satisfied on the structure. Section 3 presents two axioms of equivalence form to make every left nonnegative concatenation structure with left identity an extensive structure with identity related to an introduced operation, interprets the axioms in the context of temporal sequences, and gives the main theorem for the weighted additive model. Section 4 contains several conclusions. The proofs of the lemmas, propositions, and theorem are given in Section 5.

[1] Stationarity means that preferences are invariant over temporal sequences $(a_1, \ldots, a_n)$ under the shifts in which each component a_i is advanced or postponed by one period.

2. Basic concepts

Throughout this paper, $\mathbb{R}_0^+$ denotes the set of all nonnegative real numbers. Let $\succsim$ be a binary relation on a nonempty set A that is interpreted as a preference relation. As usual, $\succ$ denotes the asymmetric part, $\sim$ the symmetric part, and $\precsim$, $\prec$ denote reversed relations. The binary relation $\succsim$ on A is a *weak order* if and only if it is connected and transitive. Let $\cdot$ be a "partial" binary operation on A. The operation means a function from a subset B of $A \times A$ into A. The expression $a \cdot b$ is said to be *defined* (in A) if and only if $(a, b) \in B$. An element $e \in A$ denotes no change in the status quo with temporal sequences. That is, it is assumed that receiving e prior to a is no different from receiving a at present; however, ae implies advancing the receipt of a by one period, so that ae is not always $\sim a$.

In the following conditions, all the products are always assumed to be defined.

A1. Weak order: $\succsim$ is a weak order on A.
A2. Local definability: if $a \cdot b$ is defined, $a \succsim c$, and $b \succsim d$, then $c \cdot d$ is defined.
A3. Monotonicity: $a \succsim b \Leftrightarrow a \cdot x \succsim b \cdot x \Leftrightarrow x \cdot a \succsim x \cdot b$ for all $a, b, x \in A$.
A4. Left identity: e is a *left identity element*; that is, $e \cdot a \sim a$ for all $a \in A$.

The system $\langle A, \succsim, \cdot \rangle$ is a *concatenation structure* if and only if A1–A3 are satisfied. If, in addition, A4 holds, then $\langle A, \succsim, \cdot, e \rangle$ is said to be a *concatenation structure with left identity*. Throughout the paper, the trivial case where A has just a single element e is always excluded.

We now state a terminology important to this paper. An element a of a concatenation structure A is *r-nonnegative*, *l-nonnegative*, or *nonnegative* according as $x \cdot a \succsim x$, $a \cdot x \succsim x$, or both hold for all (x, a) or $(a, x) \in B$. Similarly, *r-positive*, *l-positive*, and *positive* elements can be defined by replacing $\succsim$ with the strict preference relation $\succ$. A concatenation structure is called r-nonnegative if all of its elements are r-nonnegative, and so on.

Fundamental conditions for concatenation structures are listed below.

A5. R-nonnegativity: whenever $x \cdot a$ is defined, then $x \cdot a \succsim x$.
A6. Left solvability: whenever $a \succ b$, there exists $x \in A$ such that $x \cdot b$ is defined and $a \sim x \cdot b$.

Axiom A5 is defined as the "right sided" concept, whereas A6 is defined as the "left sided" concept. That is, r-nonnegativity is the nonnegativity condition that is satisfied only for right-concatenation by a. Left solvability is a generalized solvability in the sense that only the existence of a left solution is permissible. If a concatenation structure contains a left identity element e, then by A3, a is l-positive (or l-nonnegative) if and only if $a \succ e$ (or $a \succsim e$), whereas $a \succ e$ is not always r-positive nor even r-nonnegative (see Example 1). However, the following holds.

Proposition 1. *Let $\langle A, \succsim, \cdot, e \rangle$ be a concatenation structure with left identity. If A is r-nonnegative, then $a \succsim e$ for all $a \in A$.*

Since, in A6, x is uniquely determined up to $\sim$ by A3, we write $x \sim a/b$, and $a/a \sim e$ because $a \sim e \cdot a$. Thus a partial binary operation $/$ is defined on A, which is called a *right division*. Indeed, $/$ is a function from the subset $\{(a, b) \in A \times A \,|\, a \succsim b,\ (x, b) \in B \text{ for some } x \in A\}$ into A. It may be suitable to refer to A6 as *right divisibility*.

Proposition 2. *Let $\langle A, \succsim, \cdot, e \rangle$ be a concatenation structure with left identity. If* A6 *holds, then for all $a, b, x \in A$, the following properties hold:*

(i) $(a \cdot b)/b \sim a \sim (a/b) \cdot b$ *whenever $a \cdot b$ is defined and $a \succsim b$.*

(ii) *Monotonicity of right division (*Demko, 2001*, Lemma* 3.1*):*

$$a \succsim b \Leftrightarrow a/x \succsim b/x \quad \text{whenever } a,\ b \succsim x,$$
$$a \succsim b \Leftrightarrow x/a \precsim x/b \quad \text{whenever } x \succsim a,\ b.$$

A relaxed version (Iseki, 1951) of the Archimedean property is provided. In this regard, we will inductively define the nth "left" multiple of an element a by $a^0 = e,\ a^1 = a$ and

$a^n = a \cdot a^{n-1}$ if the right-hand is defined
a^n is undefined otherwise.

A concatenation structure is said to be *left Archimedean* if every bounded sequence $\{a^n\}$ with $a \succ e$ constructed as above is finite.

A7. Left Archimedean: every bounded sequence $\{a^n\}$ consisting of the left multiples of $a \succ e$ is finite.

Definition 1. A *left nonnegative concatenation structure with left identity* is a concatenation structure $\langle A, \succsim, \cdot, e\rangle$ with left identity for which axioms A5–A7 are satisfied.

Example 1. We define a binary operation $\oplus$ on the set $\mathbb{R}_0^+$ by

$a \oplus b = \alpha a + b$ for some $\alpha > 0$.

The set $\mathbb{R}_0^+$ with this operation and the usual order $\geqslant$ is a concatenation structure with a left identity element 0. Since

$\alpha < 1 \Rightarrow a \oplus 0 < a$ for all $a \in \mathbb{R}_0^+$,
$\alpha \geqslant 1 \Rightarrow a \oplus 0 \geqslant a$ for all $a \in \mathbb{R}_0^+$,

it turns out that $\mathbb{R}_0^+$ is r-nonnegative for $\alpha \geqslant 1$, but not for $\alpha < 1$. Clearly, in both cases of $\alpha \geqslant 1$ and <1, the left Archimedean property A7 holds. Further, in both cases, it is seen that if $a > b$, then $x = (a-b)/\alpha$ is a left solution to $x \oplus b = a$. Hence $\mathbb{R}_0^+$ is a left nonnegative concatenation structure with left identity for $\alpha \geqslant 1$, but not for $\alpha < 1$.

It is worthwhile to recall a generalized concept of an extensive structure: a *PCS* (Luce et al., 1990) is a concatenation structure $\langle A, \succsim, \cdot\rangle$ that is positive and for which A6′ (a weaker type of A6; given below) and A7 are satisfied for both left- and right-concatenations.

A6′ Restricted left solvability: whenever $a \succ b$, there exists $x \in A$ such that $x \cdot b$ is defined and $a \succsim x \cdot b \succ b$.

In case the operation $\cdot$ is weakly associative (see below), it has been shown (Matsushita, 2010) that A6′ is turned into A6 under the assumption that A is Dedekind complete and continuous. However, it is very difficult to show this in the non-associative case. Note that in a left nonnegative concatenation structure with left identity A, every element $a \succ e$ is r-positive. Indeed, since $x \cdot e \succsim x$ by A5, it follows from A3 that $x \cdot a \succ x$. However, the left identity e is not always r-positive. Indeed, it can be valid that $x \cdot e \sim x$ for some x not equivalent to e (which implies that $y \cdot e \sim y$ for all $y \in A$ in the presence of A9 below). Thus, the existence of e calls for r-nonnegativity (not r-positivity) in defining our basic structure.

Remark 1. (i) Let A be a left nonnegative concatenation structure with left identity. It is seen from Proposition 1 that A consists at most of elements greater than or equal to e; as such, it is also l-nonnegative by the statement immediately before the proposition. Hence A is a nonnegative concatenation structure with left identity for which the solvability (A6) and Archimedean (A7) properties are satisfied only related to left-concatenation.

(ii) Let $A_P = \{a \in A \mid a \succ e\}$. Since $x \cdot a \succ e$ for all $x,\ a \succ e$ with $(x,\ a) \in B$ by A1, A3, and A4, $\cdot$ also turns out to be a partial binary operation on A_P. It is clear that A1–A3, A6, and A7 hold for A_P. By the above statements, A_P is l-positive and r-positive (hence positive). That is, A_P is a generalization of the PCS in the sense that the solvability and Archimedean properties are satisfied only related to left-concatenation.

We now introduce the following conditions given that all the products are defined:

- Weak associativity: $(a \cdot b) \cdot c \sim a \cdot (b \cdot c)$.
- Weak commutativity: $a \cdot b \sim b \cdot a$.

From Definition 19.3 (Luce et al., 1990), a weakly associative PCS is an *extensive structure* (see Krantz et al., 1971, for the formal definition). Following this approach, this paper defines an *extensive structure with identity* as a weakly associative concatenation structure $\langle A, \succsim, \cdot, e\rangle$ that is positive for all but elements equivalent to e and for which axioms A4, A6′, and A7 are satisfied for both left- and right-concatenations (Matsushita, 2010). Hence we see that as long as weak associativity and weak commutativity hold, a left nonnegative concatenation structure with left identity is an extensive structure with identity when it has no minimal positive element. Indeed, recall that all elements not equivalent to e (i.e., all elements greater than e) are positive (Remark 1). Obviously, A6 implies A6′if no minimal positive element exists. Finally, weak commutativity turns A4, A6, and A7 into right and left sided concepts.

Remark 2 (*Krantz et al., 1971, Theorem 3.3*)**.** If $\langle A, \succsim, \cdot\rangle$ is an extensive structure, then there exists a function u from A into the set of all positive real numbers having the following properties:

- $a \succsim b \Leftrightarrow u(a) \geq u(b)$,
- $u(a \cdot b) = u(a) + u(b)$ whenever $a \cdot b$ is defined.

Moreover, this representation u is unique up to multiplication by a positive constant.

An *additive representation* on A is a real-valued function satisfying the order-preserving and additivity properties.

3. Weighted additive model

3.1. Expression of temporal sequences

Henceforth, assume that a left nonnegative concatenation structure $\langle A, \succsim, \cdot, e\rangle$ with left identity has no minimal positive element. Concatenations expressed implicitly by juxtaposition are meant to bind more strongly than the right divisions so as to reduce the number of brackets in equivalences. For example, $(a \cdot b)/b$ reduces to ab/b.

We shall again deal with a decision-making problem of temporal sequences. Emphasis is placed on the fact that the right-branching fashion (i.e., concatenation on the left) has an entirely different meaning from the left-branching fashion (i.e., concatenation on the right) in expressing temporal sequences. This paper follows the left-branching fashion. Hence we always write $(\cdots((a_1a_2)a_3)\cdots a_{n-1})a_n$ to denote the outcome of receiving a_1 in period 1, a_2 in period 2,…, a_n in period n. In summary, the addition of a new commodity from the right means that it is received in the succeeding period. We also make it a rule to count each period number in a temporal sequence going back to the past. Therefore the left-branching notation implies that a person receives the last component a_n of the sequence in the latest period, the last component a_{n-1} of the first outside parenthesis in the period immediately before the latest period, …, and so on; finally, he/she receives the first component a_1, $n-1$ periods earlier. In contrast, $a_1(a_2 \cdots (a_{n-2}(a_{n-1}a_n)) \cdots)$ denotes the outcome of receiving $a_1, a_2, \ldots, a_{n-1}$ in the period immediately before the latest period and a_n in the latest period. Using the binary operation in Example 1, this is exemplified as follows:

$$a_1 \oplus (a_2 \oplus \cdots (a_{n-2} \oplus (a_{n-1} \oplus a_n)) \cdots) \\ = \alpha(a_1 + a_2 + \cdots + a_{n-1}) + a_n.$$

For a detailed explanation, the notation $a_r^{(1,n)} = a_1(a_2 \cdots (a_{n-2}(a_{n-1}a_n)) \cdots)$ is provided. Then it follows that $a_r^{(1,n)} = a_1 \cdot a_r^{(2,n)}$. The right-hand side expresses the situation where a single commodity is concatenated by a composite commodity on the right. According to the left-branching fashion, $a_1 \cdot a_r^{(2,n)}$ turns out to be a two-period temporal sequence. Clearly, a_1 is received in the period just before the latest one, and analogously to the comment on the left branch, the last component a_n of $a_r^{(2,n)}$ is received in the latest period. Similarly, we have $a_r^{(2,n)} = a_2 \cdot a_r^{(3,n)}$. Here since the last component a_n of $a_r^{(3,n)}$ is received in the latest period, a_2 must be received in the period just before the latest one. Finally, since $a_r^{(n-1,n)} = a_{n-1} \cdot a_n$, a_{n-1} and a_n are received in the sequential two periods in turn, as required. Accordingly, the right-branching fashion has an advantage in that we can express the (simultaneous) receipt of commodities in the same period (i.e., the period just before the latest one).

According to the above-mentioned rule, attention should be paid to expressing two temporal sequences for comparison. The receiving period of the last component in each of the two temporal sequences must be defined as the latest period. For example, in comparing $(a_1a_2)a_3$ with b_2b_3, both a_3 and b_3 are to be received in period 3. Conversely, if we are to count the period number of each component in the future direction, we may continue multiplying a temporal sequence with fewer components by e from right until the number of its components is equal to that of the other temporal sequence with more components. Indeed, the expression $(b_2b_3)e$ in comparison with $(a_1a_2)a_3$ means receiving b_2 and b_3 in period 1 and 2, respectively. Thus we can describe comparisons between sequences of commodities of the last or the future periods.

Example 2. (i) The comparison between sequences of commodities of the last three and last two periods is given as

$(a_1a_2)a_3$ vs. b_2b_3.

(ii) The comparison between sequences of commodities of the future three and future two periods is given as

$(a_1a_2)a_3$ vs. $(b_2b_3)e$.

Preferences between b_2b_3 and $(b_2b_3)e$ can vary depending on whether they are impatient (advancing the timing of future satisfaction is preferable). In case of impatience, $(b_2b_3)e \succ b_2b_3$; otherwise $(b_2b_3)e \precsim b_2b_3$.

As shown in Example 2, our approach enables us to compare temporal sequences with distinct number of periods, say m-period and n-period ($m \neq n$) sequences, because it uses multiplication to express these sequences. On the other hand, the previous approaches (Fishburn, 1970; Krantz et al., 1971) can compare only temporal sequences with the same number of periods, say n-period sequences, because their models were obtained as a variant of the additive utility on the product of n identical sets. This is a distinct advantage of our approach over the previous ones.

3.2. Axioms and the representation theorem

The following conditions are needed to construct our weighted additive utility:

A8. Weak associative-commutativity: whenever either of $a(bc)$ or $b(ac)$ is defined, the other expression is also defined and $a(bc) \sim b(ac)$.

A9. Consistent advance: whenever either of $(ab)e$ or $(ae)(be)$ is defined, the other expression is also defined and $(ab)e \sim (ae)(be)$.

For any $a \in A$, we denote the mappings of a subset of A into A defined by the rules $R_a(x) = xa$ and $L_a(x) = ax$ by R_a and L_a, respectively. In view of the comment on the right-branching fashion, A8 implies that given the same commodity c in the latest period, the preferences are invariant regarding the order of receiving a and b in the preceding period. As will be seen (in the proof of Lemma 1), this axiom plays a key role in making a new operation (defined below) commutative and associative. Further, A9 provides consistency in the meaning of multiplication by e from right, because it will be rational to consider advancing the receipt of the temporal sequence ab by one period equivalent to the temporal sequence $(ae)(be)$ in which each receipt of a and b is advanced by one period. Mathematically, this axiom implies that the right multiplication by e is a homomorphism of a subset of A into A: $R_e(ab) \sim R_e(a)R_e(b)$. Since both axioms are written as indifference to concatenations, checking them empirically seems relatively simple. Indeed, this check does not involve the problem related to the ordering of the joint effect of two or more factors from the fixed levels of the other factors, which arises when empirically checking the n-factor independence (Krantz et al., 1971). Topological concepts, such as connectedness and separability (Fishburn, 1970), are not involved either, which would have been beyond the range of intuitive recognition.

Lemma 1. *Let A be a left nonnegative concatenation structure with left identity having no minimal positive element. Assume that* A8 *and* A9 *are satisfied. Then the following equivalences*[2] *are satisfied.*

$$((a/e)b/e)c \sim (a/e)((b/e)c); \tag{1}$$

$$(a/e)b \sim (b/e)a; \tag{2}$$

$$((a/e)b)e \sim (ae/e)(be). \tag{3}$$

Define a partial binary operation $\circ$ on A by

$$a \circ b = (a/e)b. \tag{4}$$

Given that $ae/e \sim a$, it is rational to interpret a/e to mean that the receipt of a is postponed by one period. Hence $a \circ b$ is regarded as the (concurrent) receipt of a and b in the latest period. Equivalences (1) and (2) specify the weak associativity and commutativity of $\circ$, respectively: $(a \circ b) \circ c \sim a \circ (b \circ c)$ and $a \circ b \sim b \circ a$. These properties seem to be suitable to the operation denoting the concatenation in the same period. Equivalence (3) implies that $R_e(a \circ b) \sim R_e(a) \circ R_e(b)$. We also comment on the domain of $\circ$. Assume that $a \in A$. Then, by Proposition 1, $a \succsim e$, so by A6, a/e exists in A. Since e belongs to A, if $(a, e) \in B$, then by A5, $ae \succsim a$, so by Proposition 2, $a \succsim a/e$. Thus we obtain from A2 that if $(a, b) \in B$, then $(a/e, b) \in B$. This implies that B is at least contained in the domain of $\circ$.

Lemma 2. *Let A be a left nonnegative concatenation structure with left identity having no minimal positive element. If* A8 *is satisfied, then $E(A) = \langle A, \succsim, \circ, e\rangle$ is an extensive structure with identity.*

Theorem 1. *Let A be a left nonnegative concatenation structure with left identity having no minimal positive element. Assume that* A8 *and* A9 *are satisfied. Then there exist a real number $\alpha \geqslant 1$*

[2] According to Matsushita (2011), these are properties that "central" r-naturally fully ordered groupoids with left identity should satisfy.

and a function $u : A \to \mathbb{R}_0^+$ such that

(i) $a \succsim b \Leftrightarrow u(a) \geqslant u(b)$,
(ii) $u(ab) = \alpha u(a) + u(b)$ *whenever* $(a, b) \in B$,
(iii) $u(e) = 0$.

Moreover, another real number $\alpha' \geqslant 1$ and function u' satisfy (i)–(iii) *if and only if $\alpha' = \alpha$ and $u' = \gamma u$ for some real number $\gamma > 0$.*

Example 3. From (i) and (ii) of the theorem we obtain the utility representation for a preference between each pair of temporal sequences in Example 2(i) and (ii):

(i) $(a_1a_2)a_3 \succsim b_2b_3 \Leftrightarrow \alpha^2u(a_1)+\alpha u(a_2)+u(a_3) \geqslant \alpha u(b_2)+u(b_3)$.
(ii) $(a_1a_2)a_3 \succsim (b_2b_3)e \Leftrightarrow \alpha^2u(a_1) + \alpha u(a_2) + u(a_3) \geqslant \alpha^2u(b_2) + \alpha u(b_3)$.

Since $\alpha \geqslant 1$, it follows that $u((b_2b_3)e) \geqslant u(b_2b_3)$. Hence the utility model of Theorem 1 reflects impatience.

The hypothesis of the following corollaries is that A is a left nonnegative concatenation structure with left identity having no minimal positive element and for which A8 and A9 are satisfied.

Corollary 1. *If $A = \mathbb{R}_0^+$ and if $\succsim$ and $\circ$ equal the usual order $\geqslant$ and addition $+$, respectively, then $ab = \alpha a + b$ $(\alpha \geqslant 1)$ for all $a, b \in \mathbb{R}_0^+$.*

Corollary 2. *If e is a two-sided identity, then A is an extensive structure with identity.*

4. Conclusion

This paper axiomatized the weighted additive model, considering it to be a representation on generalized extensive structures. The concept of a left nonnegative concatenation structure with left identity was introduced. This structure has two partial binary operations, multiplication and right division, and its left identity e has an important meaning: division of a commodity by e implies postponing its receipt by one period. Using the division by e, a new partial binary operation was given. Then two axioms – A8 and A9 – were provided so as to make the left nonnegative concatenation structure with left identity an extensive structure with identity with respect to the newly defined operation. Finally, the weighted additive model was derived from an additive representation on the extensive structure. This enables us to compare the m-period and n-period temporal sequences where $m \neq n$. Moreover, since these two axioms are written as equivalences between concatenations, it seems to be comparatively easy to check the validity empirically. A topic for future research is generalization of the concept of left identity in the sense that it varies depending on each multiplicand.

5. Proofs

5.1. Proposition 1

Proof. Assume that $(a, a) \in B$. By A5, $aa \succsim a$, and by A1 and A4, $aa \succsim ea$. Hence by A3, we obtain $a \succsim e$ for all $a \in A$ with $(a, a) \in B$. Next, assume that $(a, a) \notin B$. Let $b \in A$ be an arbitrary element such that $(b, b) \in B$. Then b must be $\prec a$. Indeed, if not, then by A2, $(a, a) \in B$, in contradiction to the assumption. Hence, by the former case, $a \succ b \succsim e$. □

5.2. Proposition 2

Proof. (i) By definition, $(a/b)b \sim a$. Note that by A3, $x \sim a$ is a unique solution to $xb \sim ab$. Hence we obtain $ab/b \sim a$.
(ii) Let $a, b \succsim x$. Since $a \sim (a/x)x$ and $b \sim (b/x)x$ by (i), we have by A3 $a \succsim b \Leftrightarrow a/x \succsim b/x$. Let $x \succsim a, b$. Repeated use of A3 gives $a \succsim b \Leftrightarrow (x/a)a \succsim (x/a)b \Leftrightarrow x \succsim (x/a)b \Leftrightarrow (x/b)b \succsim (x/a)b \Leftrightarrow x/b \succsim x/a$. □

5.3. Lemma 1

Proof. Let $a, b \in A$, so that $a, b \succsim e$. Then by A6, a/e, b/e exist in A. Throughout the proof, all the products are assumed to be defined. Substituting a/e, b/e, e for a, b, c into both sides of the equivalence of A8 gives (2). By the same substitution except that $c = c$, we obtain

$$\begin{aligned}(a/e)((b/e)c) &\sim (b/e)((a/e)c) \quad (\text{A8})\\ &\sim (b/e)((c/e)a) \quad (2)\\ &\sim (c/e)((b/e)a) \quad (\text{A8})\\ &\sim (c/e)((a/e)b) \quad (2)\\ &\sim ((a/e)b/e)c, \quad (2)\end{aligned}$$

which proves (1). Substituting a/e for a into the equivalence of A9, we also obtain (3). □

5.4. Lemma 2

Proof. Axiom A1 is obvious for $E(A)$. As was stated immediately after (4), $E(A)$ is weakly associative and commutative. Hence from the statement before Remark 2 we may prove that A2–A7 hold for $E(A)$. Recall here that by virtue of commutativity, the validity of each axiom may be shown in either the left or right sided manner. Throughout the proof, all the products are assumed to be defined.

A2 for $E(A)$. Recall that if $a \succsim e$, then $a/e \in A$ (see the proof of Lemma 1). By (ii) of Proposition 2, we have $a \succsim c \Leftrightarrow a/e \succsim c/e$. If $b \succsim d$, then by A2 relating to $\cdot$, $(a/e, b) \in B \Rightarrow (c/e, d) \in B$.

A3 for $E(A)$. By A3 relating to $\cdot$ and (ii) of Proposition 2, we have

$$a \succsim b \Leftrightarrow (a/e)c \succsim (b/e)c \quad \text{and} \quad a \succsim b \Leftrightarrow (c/e)a \succsim (c/e)b.$$

A4 for $E(A)$. Since $e/e \sim e$, it follows from A3 and A4 with respect to $\cdot$ that $e \circ a \sim a$.

A5 for $E(A)$. As was stated in the proof of A2, $x/e \in A$ whenever $x \succsim e$. If $a \succ e$, then by A3 relating to $\cdot$, we have $(x/e)a \succ (x/e)e$, or $(x/e)a \succ x$. If $a \sim e$, then $(x/e)a \sim x$. Thus, we obtain $x \circ a \succsim x$ whenever $x \circ a$ is defined.

A6 for $E(A)$. By A6 relating to $\cdot$, let $x \in A$ be such that $a \sim xb$ whenever $a \succ b$. Since $(x, b) \in B$ and $b \succsim e$, A2 relating to $\cdot$ guarantees that $(x, e) \in B$. Hence we can set $s = xe$ to obtain $a \sim (s/e)b$.

A7 for $E(A)$. By (2), we may define the nth addition of a in the left sided manner: $na = a \circ (n-1)a$ if the right-hand side is defined for $n = 2, 3, \ldots$ and $1 \cdot a = a$. By (4), we write $na = L_{a/e}^{n-1}(a)$ for $n \geqslant 1$ where $L_{a/e}^0 = L_e$. Assume to the contrary of the Archimedean property that there exists a bounded infinite sequence $\{na\}$, $a \succ e$. Since $ae \succsim a$ by A5 relating to $\cdot$, we have by Proposition 2 $a \succsim a/e$. Similarly, $a/e \succ e$ from the inequality $a \succ e$. Since the mapping $L_{a/e}^{n-1}$ is order preserving, it follows that $na = L_{a/e}^{n-1}(a) \succsim L_{a/e}^{n-1}(a/e)$. This implies the existence of a bounded infinite sequence $\{(a/e)^n\}$, which contradicts the left Archimedean property relating to $\cdot$. □

5.5. Theorem 1

We first mention the concepts of "partial substructure" (Ježek, 2008; Krantz et al., 1971, Theorem 3.5) and "order-isomorphism". Let $\langle A, \succsim, \circ, e\rangle$ be an extensive structure with identity, and let S be a nonempty subset of A. A relation is defined on S by the restriction of $\succsim$ to S. A partial binary operation is defined by the restriction of $\circ$ to S such that $a \circ b$ $(a, b \in S)$ is defined in A and belongs to S. Then S is said to be an *extensive substructure with identity of A* if it contains the identity e and is an extensive structure with respect to the relation and operation defined above. Hereafter, we will denote this relation and operation on S by the same symbols $\succsim$ and $\circ$, respectively. Extensive structures A and A' with identity are *order-isomorphic* if there exists an onto mapping $\iota : A \to A'$ having the order-preserving and homomorphic properties. (Here the order-preserving property implies that ι is a one-to-one mapping up to $\sim$.)

Proof. Since $E(A)$ is an extensive structure with identity by Lemma 2, it is seen from Remark 2 that there exists an additive representation u on $E(A)$. Then since $u(a) = u(e \circ a) = u(e) + u(a)$, we obtain $u(e) = 0$. In view of A2, the hypothesis $(a, b) \in B$ implies that $(a, e) \in B$. Since $ab \sim ae \circ b$, $u(ab) = u(ae) + u(b)$. To construct the weighted additive model, it suffices to show that $u(ae) = \alpha u(a)$ for some $\alpha \geqslant 1$. In this regard, the following lemma is provided.

Lemma 3. *Let* $A_e = \{ae \,|\, a \in A, (a, e) \in B\}$ *and* $A' = \{a \in A \,|\, (a, e) \in B\}$. *Let* $E(A_e) = \langle A_e, \succsim, \circ, e\rangle$ *and* $E(A') = \langle A', \succsim, \circ, e\rangle$. *Then both* $E(A_e)$ *and* $E(A')$ *are extensive substructures with identity of* $E(A)$, *and are order-isomorphic.*

Proof. Since $ee \sim e$, e is an identity element of A_e with respect to the operation $\circ$ defined above. Obviously, the order $\succsim$ defined above is a weak order on A_e. Equivalence (3) is rewritten as

$$ae \circ be \sim (a \circ b)e. \tag{5}$$

Assume that $ae \circ be$ is defined in A. It then follows from (5) that $ae \circ be \in A_e$. Hence it is seen that A2–A7 hold for $E(A_e)$. We will prove only A6 because the other axioms are obvious. Assume that $ae \succ be$. From the proof of A6 in Lemma 2, there exists $x \in A$ such that $ae \sim x(be)$ and $(x, e) \in B$. Since $(xe) \circ (be) \sim (xe/e)(be)$, we obtain $ae \sim (xe) \circ (be)$, as required. In comparison to A9, recall that the presupposition of equivalence (5) is that $ae \circ be$ is defined in A if and only if $a \circ b$ is defined in A and $(a \circ b, e) \in B$. Hence it is easily verified that A2–A7 hold for $E(A')$. Thus we conclude that $E(A_e)$ is an extensive structure with respect to the above defined $\succsim$ and $\circ$ if and only if $E(A')$ is. This proves the former assertion. For the latter assertion, define a mapping ι of A' to A_e by $\iota(a) = ae$. By definition, ι is an onto mapping. By A3, $a \succsim b \Leftrightarrow \iota(a) \succsim \iota(b)$, and by (5), $\iota(a \circ b) \sim \iota(a) \circ \iota(b)$. Thus we obtain that $E(A_e)$ and $E(A')$ are order-isomorphic. $\square$

From this lemma it is seen that the restriction of u is an additive representation on $E(A_e)$. Define $u_e(a) = u(\iota(a))$ for all $a \in A'$. Clearly, u_e is order-preserving on A'. Since

$$\begin{aligned} u_e(a \circ b) &= u(\iota(a \circ b)) = u(\iota(a) \circ \iota(b)) \\ &= u(\iota(a)) + u(\iota(b)) = u_e(a) + u_e(b), \end{aligned}$$

it follows that u_e is an additive representation on $E(A')$. Hence by the uniqueness assertion of Remark 2, there is a positive real number α such that $u_e(a) = \alpha u(a)$ (because u is also an additive representation on $E(A')$). Moreover, since $a \precsim ae$ for all $a \in A$ with $(a, e) \in B$ by A5, $u(a) \leqslant u(ae) = \alpha u(a)$. Thus $\alpha \geqslant 1$. Finally, we prove the uniqueness assertion. Assume that α' and u' satisfy (i)–(iii). Then since u' is an additive representation on $E(A)$, by the uniqueness assertion, we have $u' = \gamma u$ for some $\gamma > 0$. However, since $u_e(a) = u(ae)$ and $u'_e(a) = u'(ae)$, $u'_e = \gamma u_e$ must be valid, and hence $\alpha' u' = \gamma \alpha u$. Eliminating u from the equations $u' = \gamma u$, $\alpha' u' = \gamma \alpha u$ and noting that the resulting equation holds for all $a \in A$ with $(a, e) \in B$, we have $\alpha' - \alpha = 0$, or $\alpha' = \alpha$, as required. $\square$

5.6. Corollary 1

Proof. It suffices to show that function u in the proof of Theorem 1 is strictly increasing. Indeed, if so, then since u is additive and strictly monotonic on $\mathbb{R}_0^+$, it is well known (Falmagne, 1985, Theorem 3.2) that $u(a) = sa$ for some $s \in \mathbb{R}$. Setting $s = 1$, we obtain $ab = \alpha a + b$. To prove that u is strictly increasing, assume that $a > b$. By A6, $a = xb$ for some $x \in A$ with $x \neq 0$. Hence $u(a) = u(x) + u(b) > u(b)$, as required. $\square$

5.7. Corollary 2

Proof. Since $a/e \sim a$, equivalences (1) and (2) reduce to $(ab)c \sim a(bc)$ and $ab \sim ba$, respectively. By the statement immediately before Remark 2, A is an extensive structure with identity. $\square$

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